Four times chaos

After quite some hiatus, the analog computer application notes are back with more chaotic systems, a topic I am quite fond of. :-) Four such systems are described in the following. The first three systems require three multipliers and thus need THE ANALOG THINGs while the last system only needs one multiplier and runs on a single THE ANALOG THING.

1 Halvorsen attractor

The HALVORSEN *attractor* was proposed by ARNE DEHLI HALVORSEN and exhibits an incredibly symmetry with respect to its defining equations and the result phase space plots:[1]

$$\begin{aligned}\dot{x} &= -ax - 4y - 4z - y^2\\ \dot{y} &= -ay - 4z - 4x - z^2\\ \dot{z} &= -az - 4x - 4y - x^2\end{aligned}$$

Due to this underlying symmetry, a single scaling factor λ fits all three variables x, y, and z. A quick numerical simulation shows that $\lambda = \frac{1}{15}$ is a good choice, yielding the following scaled system:

$$\begin{aligned}\dot{x} &= -ax - 4y - 4z - 15y^2\\ \dot{y} &= -ay - 4z - 4x - 15z^2\\ \dot{z} &= -az - 4x - 4y - 15x^2\end{aligned}$$

Since a coefficient of 15 is a bit out of range for a classic analog computer, it is advisable to downscale the system again with a factor of $\frac{1}{10}$:

$$\begin{aligned}\dot{x} &= -\frac{a}{10}x - 0.4y - 0.4z - 1.5y^2\\ \dot{y} &= -\frac{a}{10}y - 0.4z - 0.4x - 1.5z^2\\ \dot{z} &= -\frac{a}{10}z - 0.4x - 0.4y - 1.5x^2\end{aligned}$$

[1] The parameter is $a = 1.89$.

This system can now be implemented directly as shown in figure 1. This system is highly sensitive with respect to the initial conditions $x(0)$, $y(0)$, and $z(0)$. It is simple to "fall" into a fixed point instead of generating the beautiful attractor shown in figure 2.[2] Due to the symmetry in the defining equations, a phase space plot x vs. y looks basically identical to other combinations, so a single plot is sufficient.

[2]The initial conditions $x(0) = y(0) = z(0) = 0.3$ work well.

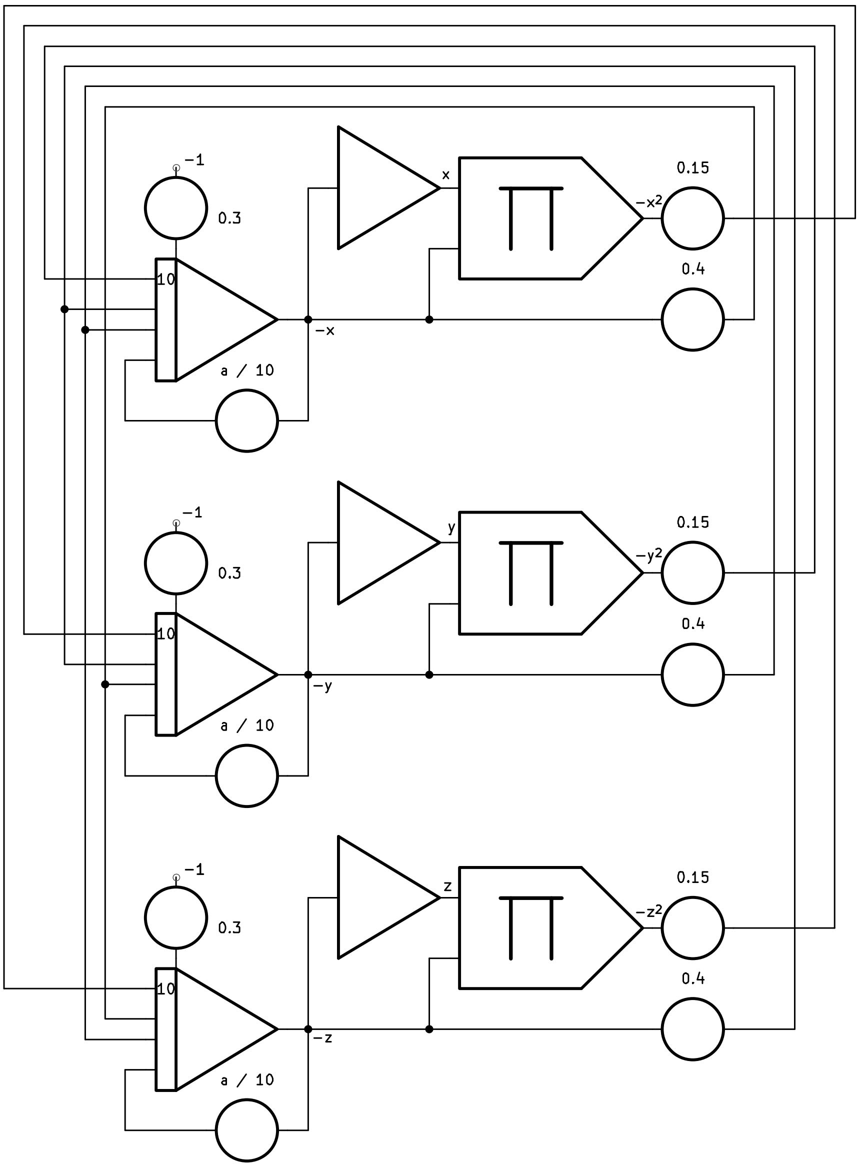

Figure 1: Analog computer setup for the HALVORSEN attractor

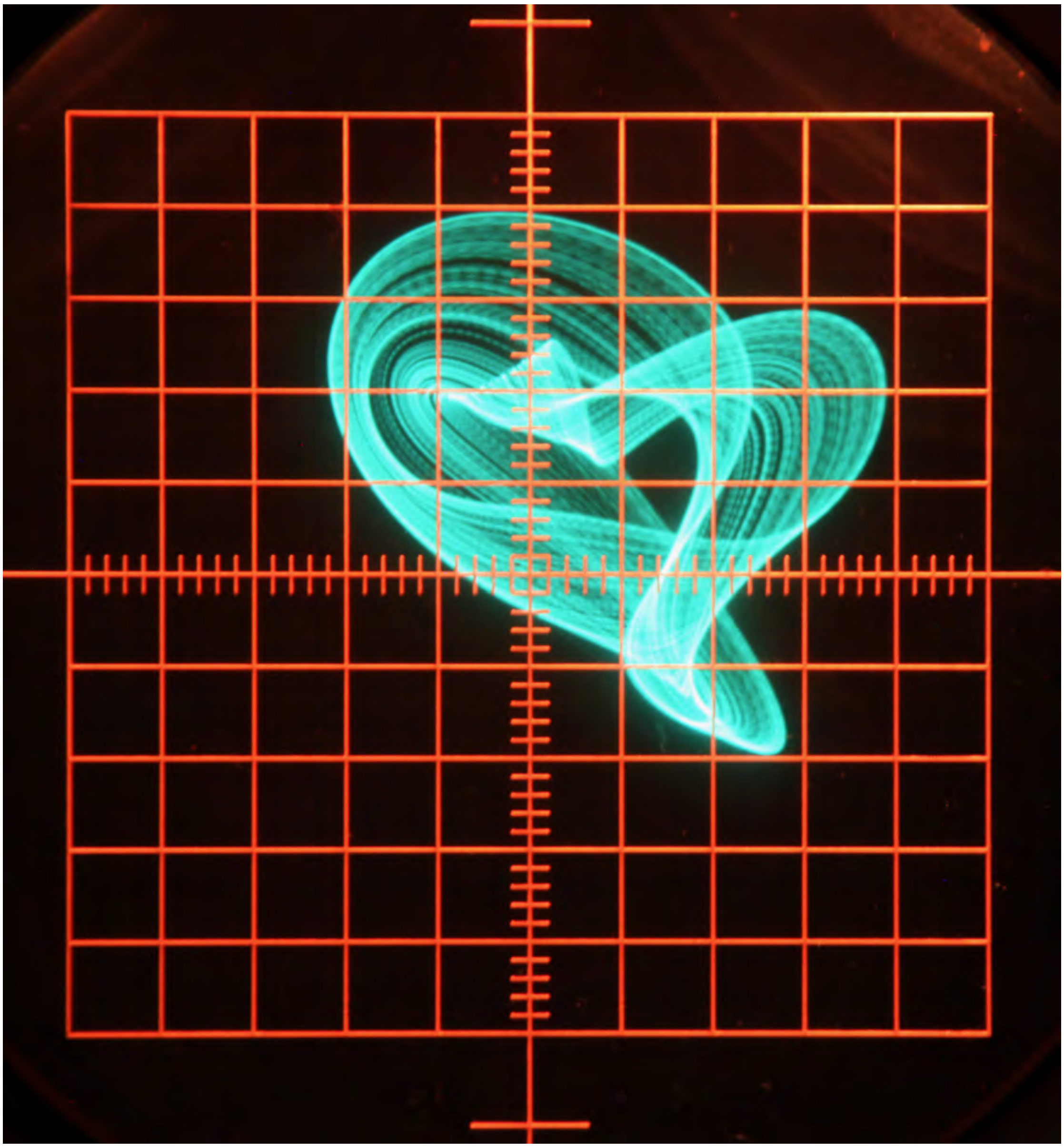

Figure 2: Phase space plot of the HALVORSEN system

2 Dadras attractor

The DADRAS attractor was first described in [DADRAS et al. 2009] and is defined by the following set of coupled DEQs

$$\begin{aligned}\dot{x} &= y - ax + byz\\ \dot{y} &= cy - xz + z\\ \dot{z} &= dxy - hz\end{aligned}$$

with $a = 3$, $b = 2.7$, $c = 4.7$, $d = 2$, and $h = 9$. The value $c = 1.7$ yields an even nicer result, so in the following this will be used.

Suitable scaling factors are $\lambda_x = \frac{1}{20}$, $\lambda_y = \frac{1}{15}$, and $\lambda_z = \frac{1}{20}$. Since there are quite some products which must be scaled, chosing $\lambda_y = \frac{1}{20}$ instead of $\frac{1}{15}$ has been chosen so that more factors cancel out during the scaling process. The resulting system looks like this:

$$\begin{aligned}\dot{x} &= y - 3x + 54yz\\ \dot{y} &= 1.7y - 20xz + z\\ \dot{z} &= 40xy - 9z\end{aligned}$$

Since this still has some very large coefficients, everything is scaled down again by a factor of $\frac{1}{10}$:

$$\begin{aligned}\dot{x} &= 0.1y - 0.3x + 5.4yz\\ \dot{y} &= 0.17y - 2xz + 0.1z\\ \dot{z} &= 4xy - 0.9z\end{aligned}$$

Good initial conditions are $-x(0) = 0.06$, $-y(0) = 0.01$, and $-z(0) = -0.01$.

The resulting analog computer setup is shown in figure 3 while figure 4 shows two phase space plots of the system.

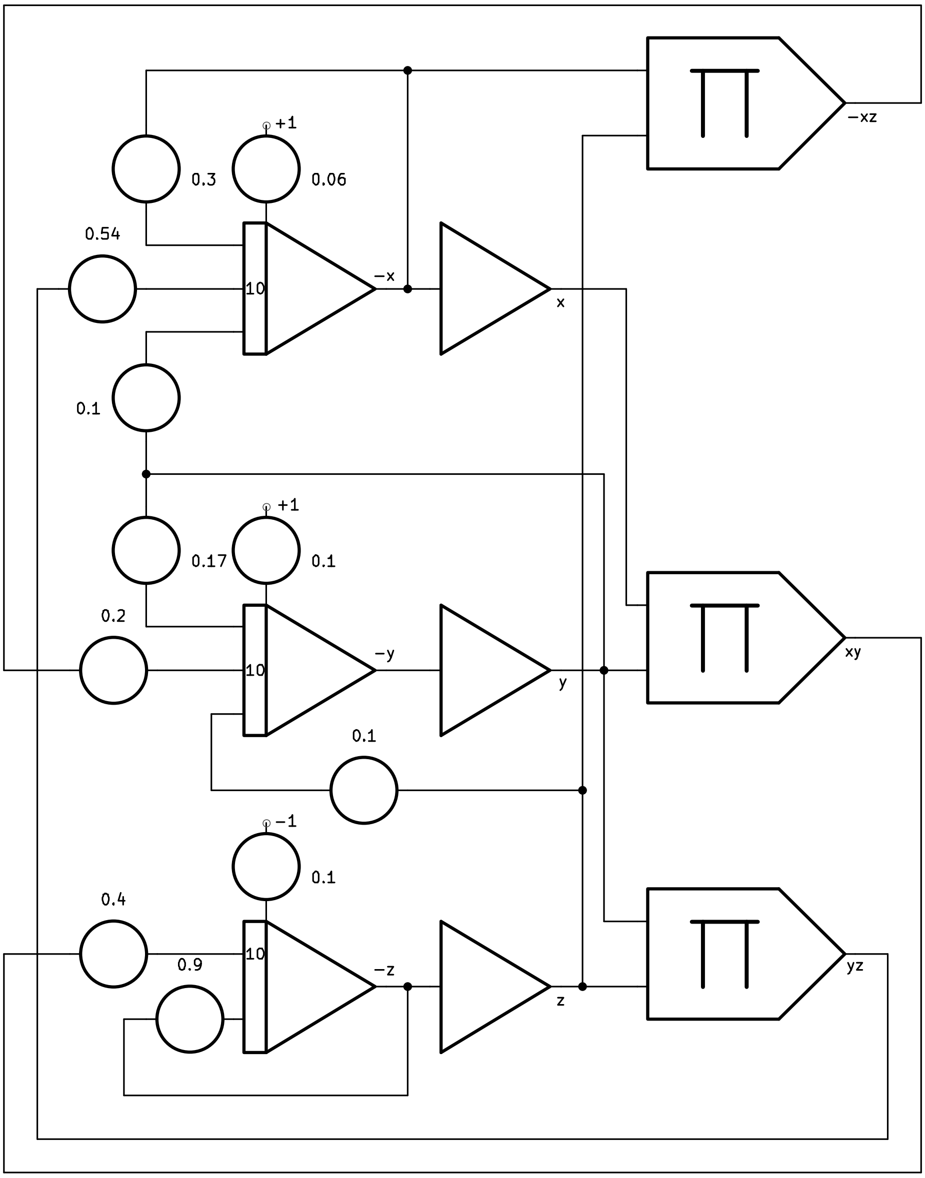

Figure 3: Analog computer setup for the DADRAS system

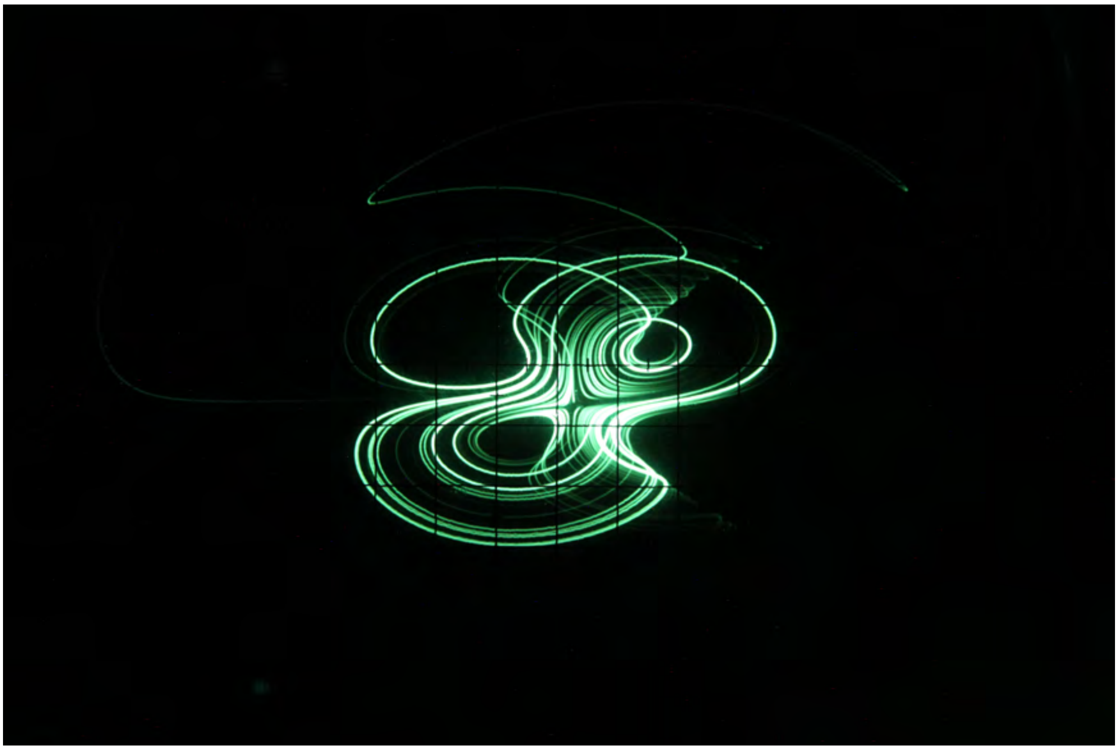

Figure 4: Phase space plot of the DADRAS system with $c = 1.7$ instead of $c = 4.7$

3 Reduced Henon-Heiles system

In 1962 MICHEL HÉNON and CARL HEILES developed a mathematical model of the non-linear motion of a star revolving around a galactic center in a plane. This system was originally defined as

$$
\begin{aligned}
\dot{x} &= p_x \\
\dot{p}_x &= -x - 2\lambda xy \\
\dot{y} &= p_y \\
\dot{p}_y &= -y - \lambda x^2 - y^2
\end{aligned}
$$

with $\lambda = 1$.[3]

A reduced version of this has been shown by [Sprott 2016, pp. 132 f.]:

$$
\begin{aligned}
\ddot{x} &= xy \\
\ddot{y} &= 0.57y^2 - x^2
\end{aligned}
$$

Interestingly, this system can be directly implemented on an analog computer as all variables, $\dot{x}, x, \dot{y}$, and y, are well within $[-1, 1]$. In fact, $\dot{x}$ and $\dot{x}$ are so small that it is worthwhile to scale these two up by a factor of 10. The system is very sensitive with respect to its initial conditions $x(0)$ and $y(0)$ so it is worthwhile to have the analog computer (in this case two coupled THE ANALOG THINGs) in repetitive mode of operation to observe the result of varying $x(0)$ and $y(0)$.

Figure 5 shows the resulting program, while figure 6 shows a typical phase space plot of the system.

[3] See [Hénon et al. 1964] and [Emanuelsson 2004] for details.

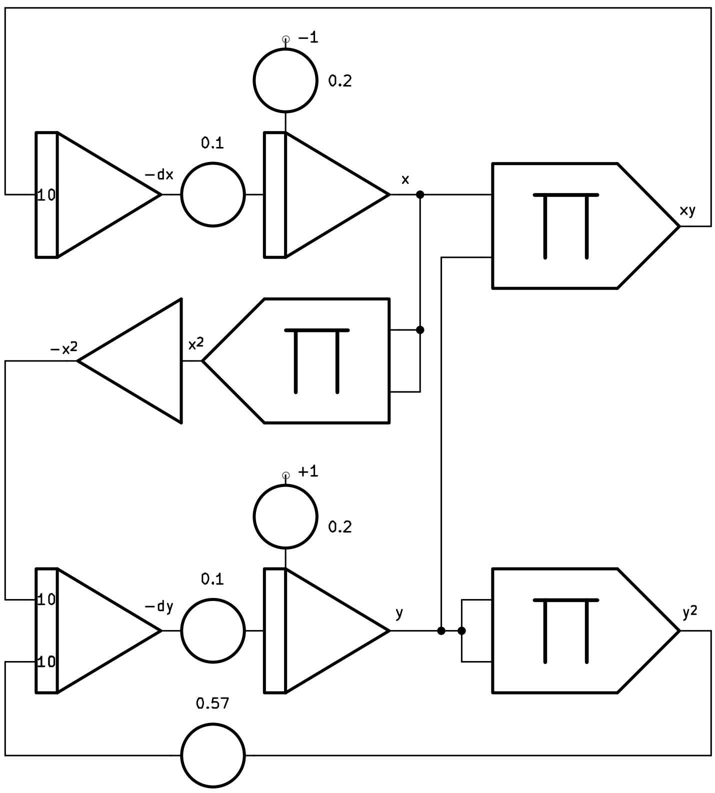

Figure 5: Analog computer setup for the reduced HENON-HEILES system

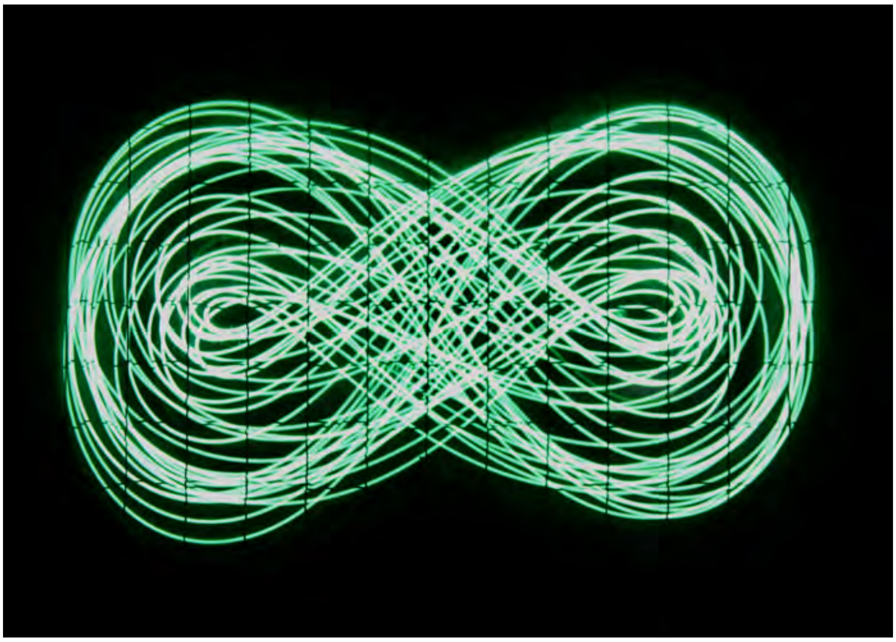

Figure 6: Phase space plot ($\dot{x}/\dot{y}$) of the reduced Hénon-Heiles system

4 Sprott SQ_F system

Another really nice and simple chaotic system is also due to SPROTT, the SQ_F system, see [SPROTT 2016, pp. 68 ff.]. This is an especially nice system as it only requires one multiplier and is thus ideally suited to be implemented on THE ANALOG THING.

This system is defined by

$$\dot{x} = y + z,$$
$$\dot{y} = -x + 0.5y \text{ and}$$
$$\dot{z} = x^2 - z$$

with a common scaling factor of $\lambda_x = \lambda_y = \lambda_z = \frac{1}{5}$, which yields the following scaled system:

$$\dot{x} = y + z,$$
$$\dot{y} = -x + 0.5y$$
$$\dot{z} = 5x^2 - z$$

Figure 7 shows the resulting analog computer setup while figure 8 shows a phase space plot of the system.

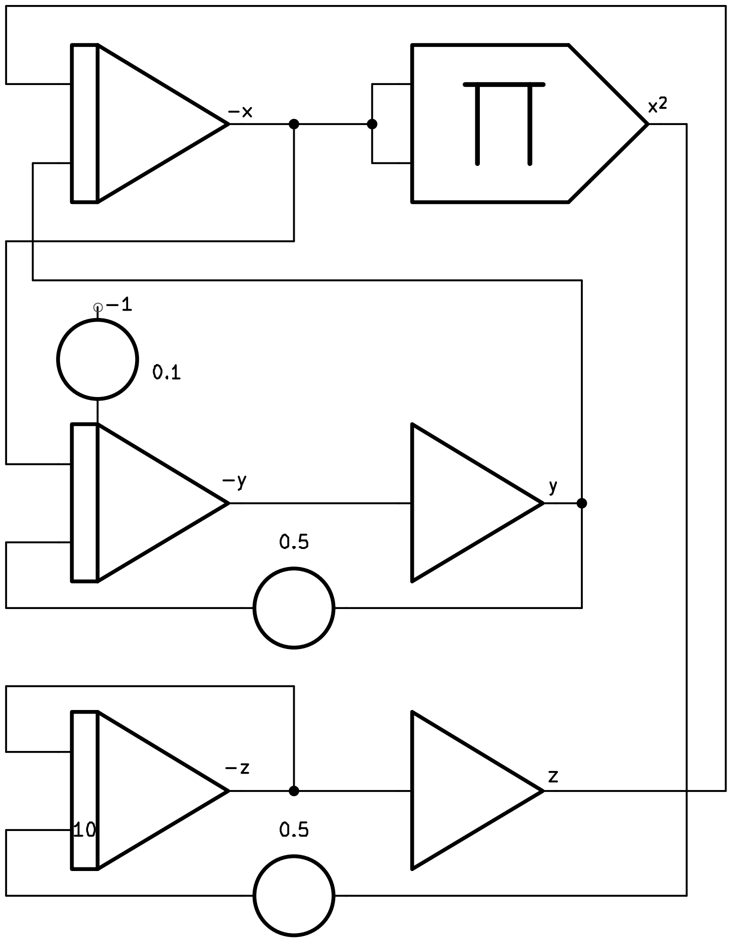

Figure 7: Analog computer setup for the SQ_F system

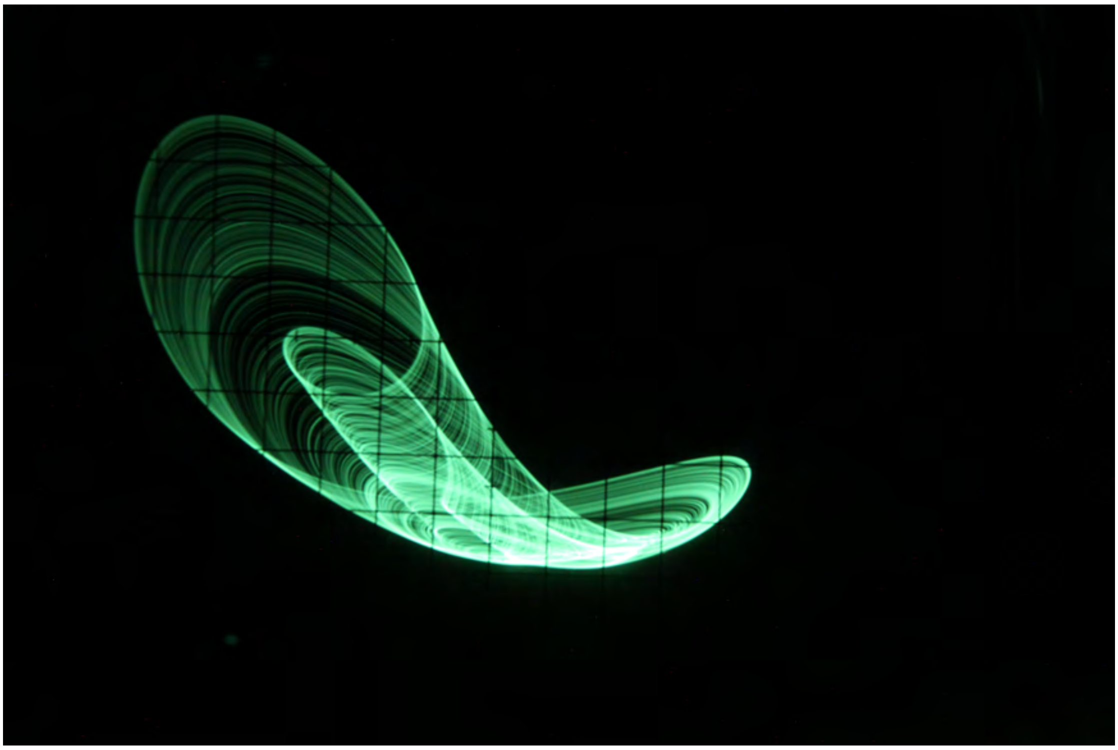

Figure 8: Phase space plot of the SQ_F system

A How to take screen shots

The question of how these screen shots are taken comes up quite often, so the setup used is described in this appendix. First of all, best results are achieved with a classic analog oscilloscope. This must not be a fancy one - it just needs a good focus. The picture of the HALVORSEN attractor was taken from a Telefunken OMS800 oscilloscope from 1966, while a Leader LBO-9C was used for the remaining pictures.

The basic setup is shown in figure 9. On the bottom is the LBO-9C x/y-display with two THATs sitting on top. An acrylic A4 book stand is perfect to place two THATs next to each other. To take good pictures some kind of a "mini photo booth" is required unless one can close all window blinds completely or wait until night time. In this case a card board box has been adapted to fit around the display on one side with a round hole for the camera lens on the other side as shown in figure 10.[4]

There is plenty of room to experiment with exposure times, beam intensity, etc. Best results were obtained with ISO 1600, an exposure time varying between one and four seconds, and the intensity turned down so far that the picture is just visible for the naked eye.

Happy analog computing! :-)

[4]Please excuse the cluttered background, my lab tends to be as chaotic as the attractors shown here. . . :-)

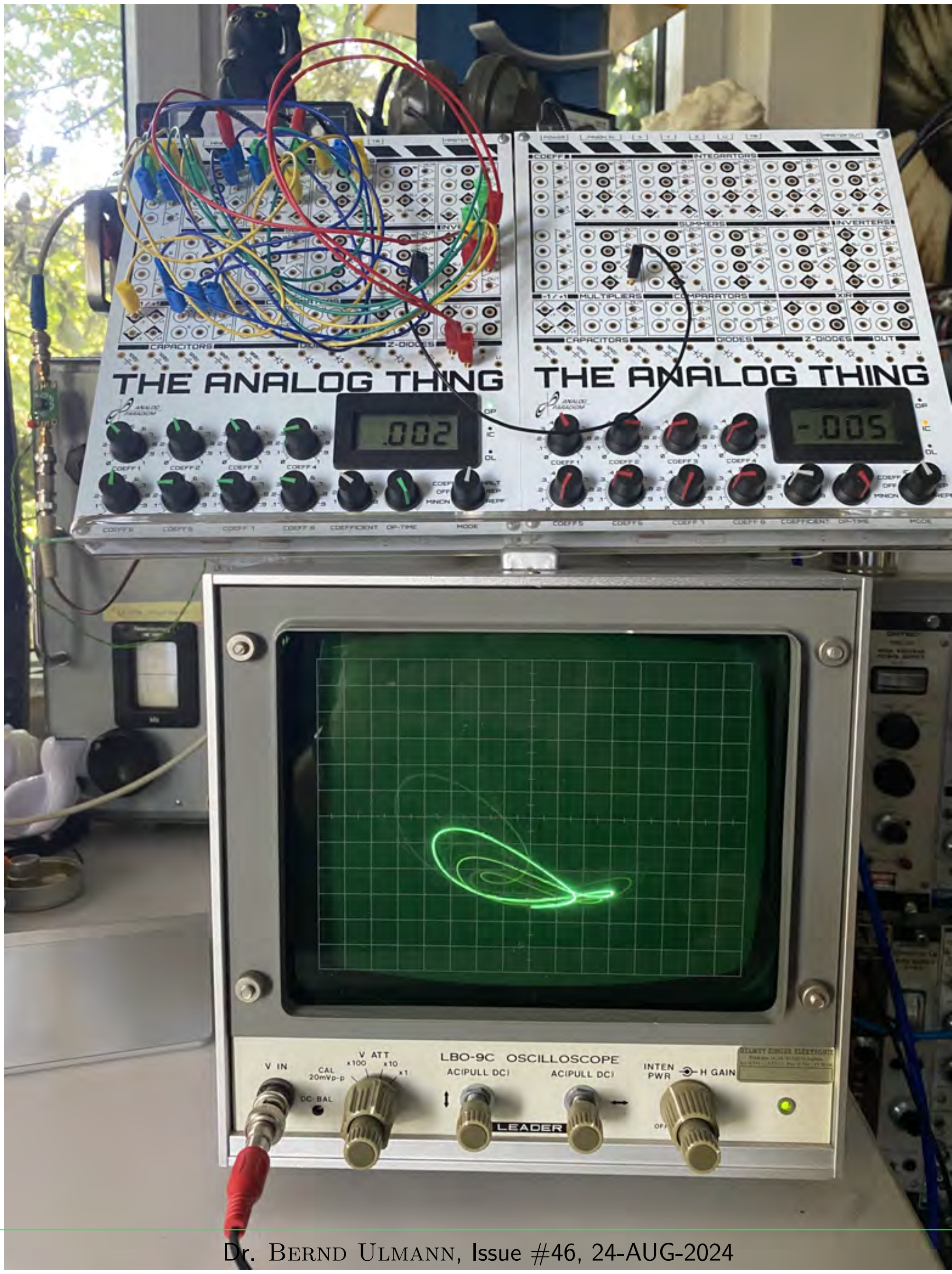

Figure 9: Basic setup

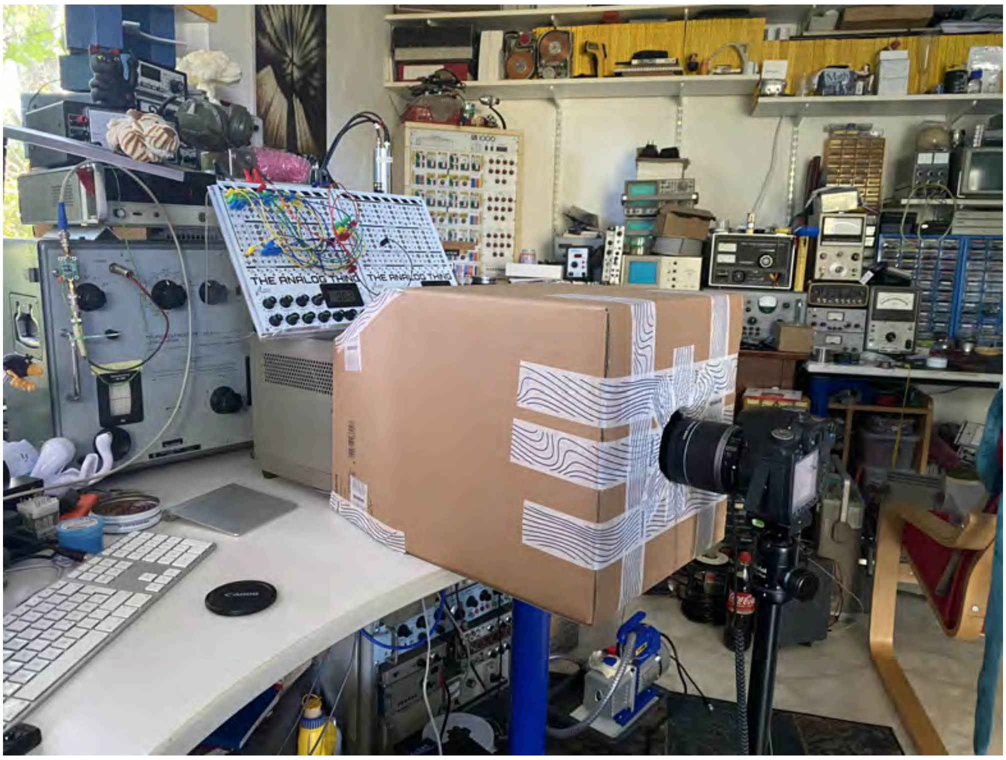

Figure 10: Card board box setup for taking screen shots

References

[Dadras et al. 2009] Sara Dadras, Hamid Reza Momeni, "A novel three-dimensional chaotic system generator two, three and four-scroll attractors", in *Physics Letters A*, Vol. 373, Issus 40, 28. September 2009, pp. 3637–3642

[Emanuelsson 2004] Christian Emanuelsson, *Chaos in the Hénon-Heiles system*, `https://jfuchs.hotell.kau.se/kurs/amek/prst/11_hehe.pdf`, retrieved 23.08.2024

[Hénon et al. 1964] Michel Hénon, Carl Heiles, "The Applicability of the Third Integral Of Motion: Some Numerical Experiments", in *The Astronomical Journal*, Vol. 69, No. 1, February 1964, pp. 73–79

[Sprott 2016] Julien Clinton Sprott, *Elegant Chaos – Algebraically Simple Chaotic Flows*, World Scientific, 2016

[Steeb 1994] Willi-Hans Steeb, *Chaos und Quantenchaos in dynamischen Systemen*, BI Wissenschaftsverlag 1994.