A passive network for solving the two-dimensional heat equation

The aim of this application note is to show how the two-dimensional heat transfer equation can be solved using a passive resistor-capacitor (RC) network. The solution of the underlying equation

$$\dot{u} = \alpha\nabla^2 u \tag{1}$$

is described in [Giles, Ulmann, 2020]. Using finite differences, equation (1) can be transformed into

$$\dot{u}_{i,j} = \alpha\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right) + q_{i,j}.$$

Applying Kirchhoff's law to the center node $u_{i,j}$ shown in figure 1 by summing all the currents into the node gives the following equation with $u_{i,j}$ denoting the voltage at node i, j

$$\frac{1}{R}\left((u_{i,j-1} - u_{i,j}) + (u_{i,j+1} - u_{i,j}) + (u_{i-1,j} - u_{i,j}) + (u_{i+1,j} - u_{i,j})\right) - C\dot{u}_{i,j} + I_{i,j} = 0.$$

The first four terms are the currents through the resistors, the fifth term is the current through the capacitor and the last term is a current injected into the node. Rearranging yields

$$C\dot{u}_{i,j} = \frac{1}{R}\left((u_{i,j-1} - u_{i,j}) + (u_{i,j+1} - u_{i,j}) + (u_{i-1,j} - u_{i,j}) + (u_{i+1,j} - u_{i,j})\right) + I_{i,j}.$$

This equation is identical to the discrete form of the heat transfer equation and it can therefore be seen that it is possible to use a passive RC network to solve the heat transfer problem, by considering

voltage ≡ temperature and

current ≡ heat flux.

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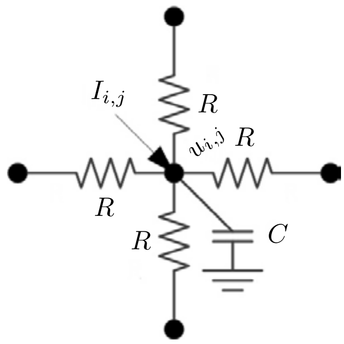

Figure 1: A single RC-node

A circuit to solve a set of equations is traditionally known as an *Electrical Analogy* or *Analog* and this approach was widely used up until the 1960s for solving a wide range of heat transfer, electromagnetic and fluid flow problems.

In this example, shown in figure 2, a plate 16×16 cm is considered; its top and bottom surfaces are insulated and its outside boundary is held at a constant temperature. A 1×1 cm grid is set up on the plate. By exploiting the symmetries along the x- and y-axes, it is only necessary to solve for one quadrant of the plate.[1]

The 8×8 RC network shown in figure 3 was designed to model one quadrant of the plate and was constructed using 180 k resistors and 100 nF capacitors. The implementation is shown in figure 4. The connection labelled $I_{0,0}$ enables current (heat flux) to be injected into, or a voltage (temperatur) to be applied to, node $u_{0,0}$. Connections α and β allow the fixed boundary voltages (temperatures) to be established. All voltages are relative to ground.

Two sets of results are presented here. Figure 5 shows the responses at the diagonal nodes $u_{i,i}$ to the injection of a pulse $I_{0,0}$ at node $u_{0,0}$, with all the nodes initially at the boundary temperature. The diffusion of the temperature across the plate can clearly

[1]Although the problem is also symmetric along the 45 degree diagonals, it is difficult to exploit this symmetry using a passive RC network, unlike the case shown in [Giles, Ulmann, 2020] where the problem was solved using an analog computer.

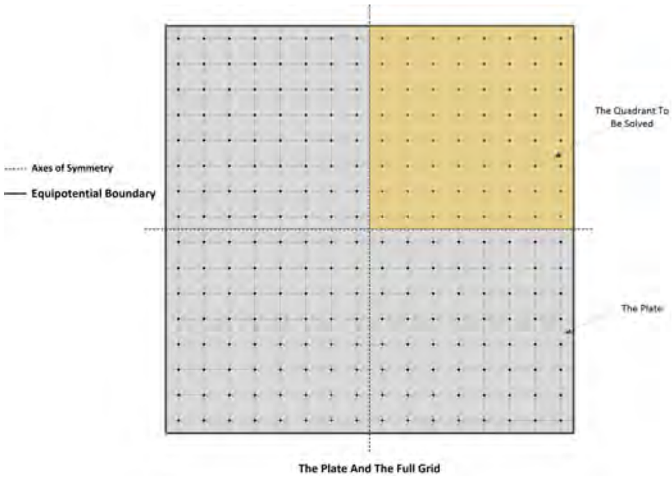

Figure 2: 16×16 cm plate structure

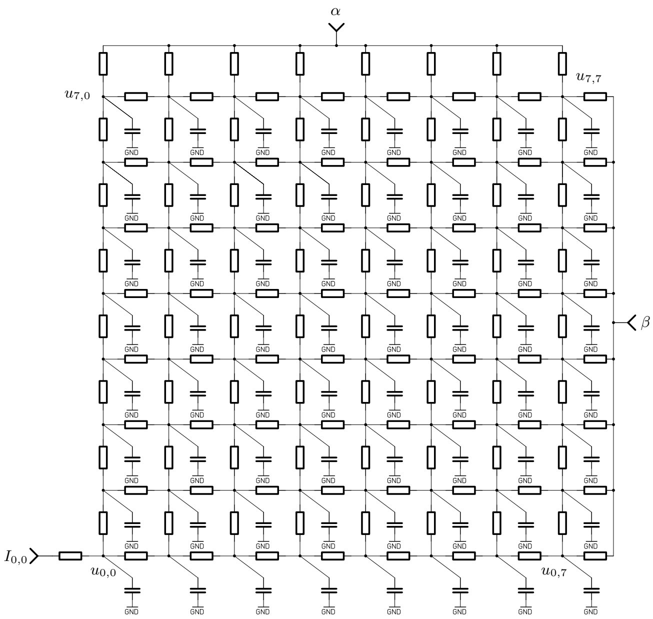

Figure 3: Passive network for one quadrant

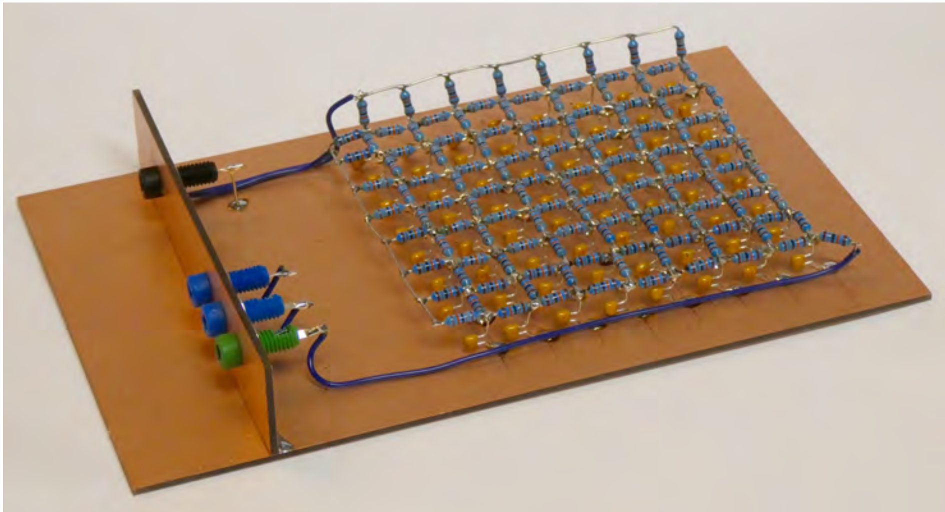

Figure 4: Implementation of the two-dimensional RC-network

be seen, together with the eventual return to the boundary condition temperature.

The graphs shown in figure 6 depict the steady state response to forcing the temperature at node $u_{0,0}$ to a fixed value and allowing the temperatures throughout the plate to reach steady state values. The picture on the left shows the temperature distribution across the plate. The graph on the right shows the heat flux densities across the plate, which were obtained by measuring the voltages across the resistors (and hence the heat flux between the associated nodes), to give the x and y components of the heat flux, and then calculating the heat flux magnitudes in between the nodes.

References

[Giles, Ulmann, 2020] Chris Giles, Bernd Ulmann, "Solving the two-dimensional heat-equation", in *Analog Computer Applications*, issue 24, http://analogparadigm.com/downloads/alpaca_24.pdf

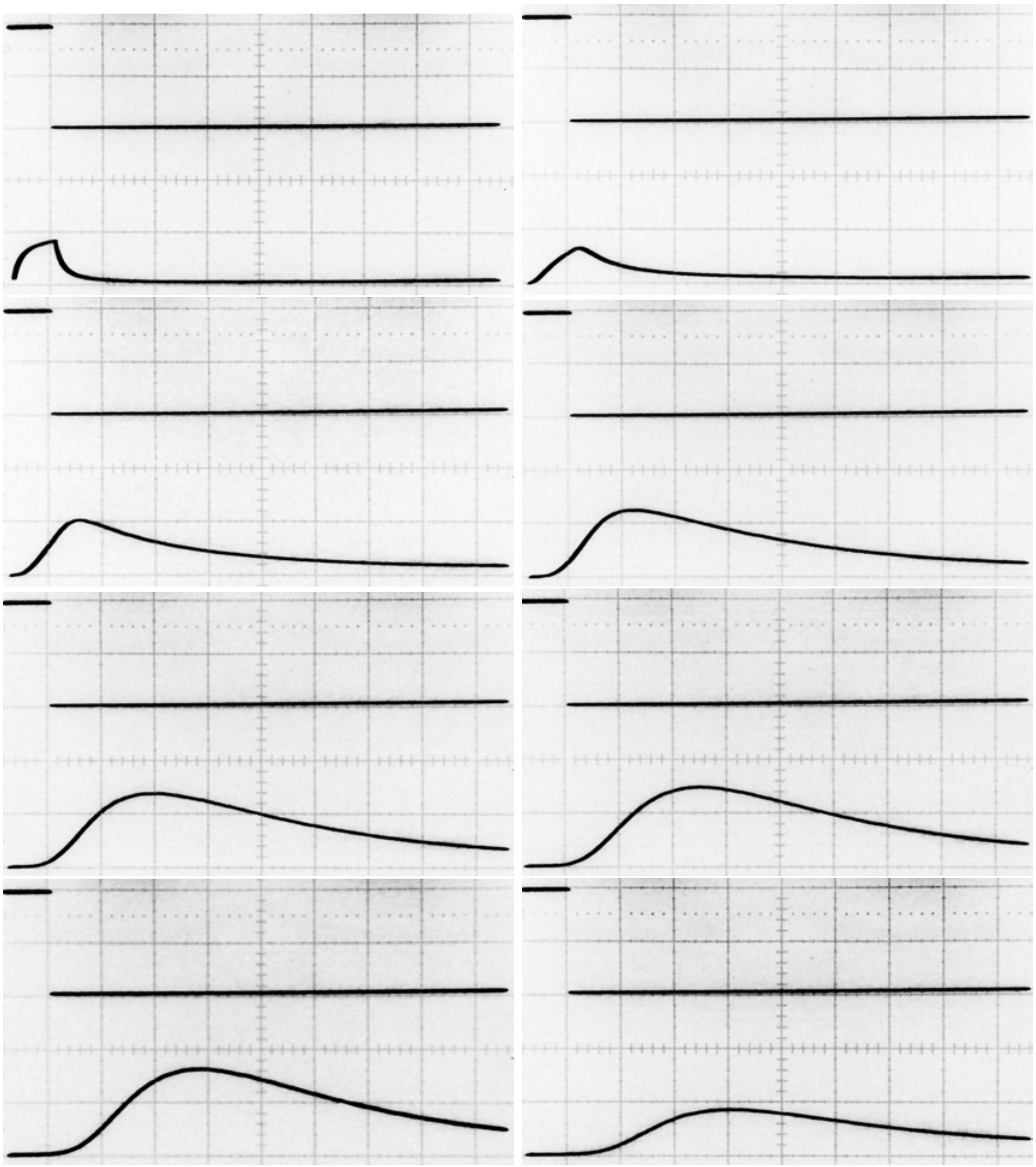

Figure 5: Response to a step input at nodes $u_{0,0}$, $u_{1,1}$, $\dots u_{7,7}$

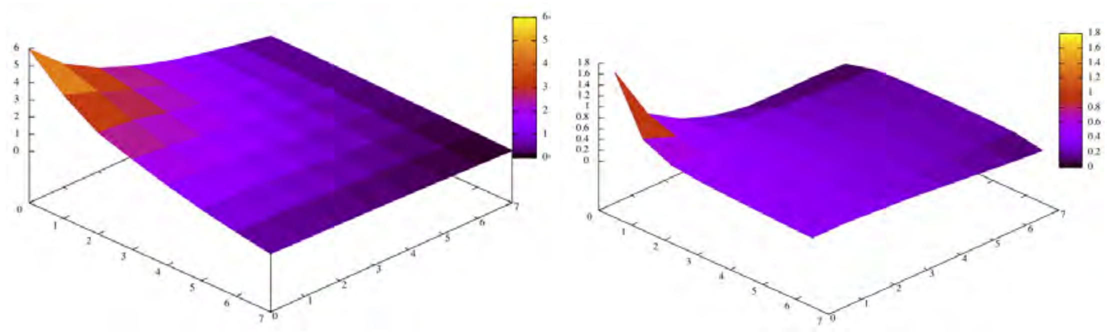

Figure 6: Steady state of the network and differences between adjacent nodes