



The Li system

In [LETELLIER et al. 2007] a LORENZ-like chaotic system, the *LI system*, is analysed in detail. This system is implemented on two THE ANALOG THINGS in this application note with a special emphasis on scaling which is a bit tricky in this particular case.

The system has the form

$$\dot{x} = a(y - x) + dxz = ay - ax + dxz \quad (1)$$

$$\dot{y} = bx + ky - xz \quad (2)$$

$$\dot{z} = cz + xy - ex^2 \quad (3)$$

with parameters $a = 40$, $b = 55$, $c = \frac{11}{6}$, $d = 0.16$, $e = 0.65$, and $k = 20$. To determine suitable scaling factors λ_x , λ_y , and λ_z a quick numerical experiment was performed which showed that $|x| \leq 150$, $|y| \leq 200$, and $|z| \leq 250$ yielding

$$\lambda_x = \frac{1}{150}$$

$$\lambda_y = \frac{1}{200}$$

$$\lambda_z = \frac{1}{250}.$$

In the first step x is scaled down by multiplying every term of the right-hand side of (1) by λ_x . To correct for this downscaling, every occurrence of x on the right-hand sides of (1), (2), and (3) must be scaled up by multiplying with $\frac{1}{\lambda_x}$, resulting in

$$\dot{x} = \lambda_x ay - ax + dxz \quad (4)$$

$$\dot{y} = \frac{1}{\lambda_x} bx + ky - \frac{1}{\lambda_x} xz \quad (5)$$

$$\dot{z} = cz + \frac{1}{\lambda_x} xy - \frac{1}{\lambda_x^2} ex^2. \quad (6)$$



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In the next step λ_y is applied to equations (4), (5), and (6) yielding

$$\dot{x} = \frac{\lambda_x}{\lambda_y}ax - ax + dxz \quad (7)$$

$$\dot{y} = \frac{\lambda_y}{\lambda_x}bx + ky - \frac{\lambda_y}{\lambda_x}xz \quad (8)$$

$$\dot{z} = cz + \frac{1}{\lambda_x\lambda_y}xy - \frac{1}{\lambda_x^2}ex^2. \quad (9)$$

Finally, the system (7), (8), (9) is scaled with λ_z yielding

$$\dot{x} = \frac{\lambda_x}{\lambda_y}ax - ax + \frac{1}{\lambda_z}dxz \quad (10)$$

$$\dot{y} = \frac{\lambda_y}{\lambda_x}bx + ky - \frac{\lambda_y}{\lambda_x\lambda_z}xz \quad (11)$$

$$\dot{z} = cz + \frac{\lambda_z}{\lambda_x\lambda_y}xy - \frac{\lambda_z}{\lambda_x^2}ex^2. \quad (12)$$

Inserting λ_x , λ_y , and λ_z into equations (10), (11), and (12) yields

$$\dot{x} \approx 1.333ay - ax + 250dxz \quad (13)$$

$$\dot{y} \approx 0.75bx + ky - 187.5xz \quad (14)$$

$$\dot{z} \approx cz + 120xy - 90ex^2 \quad (15)$$

Inserting a , b , c , d , e , and k into (13), (14), and (15) now yields

$$\dot{x} \approx 53.32y - 40x + 40xz \quad (16)$$

$$\dot{y} \approx 41.25x + 20y - 187.5xz \quad (17)$$

$$\dot{z} \approx 1.833z + 120xy - 58.5x^2. \quad (18)$$

This ensures that x , y , and z are well within the permissible interval of $[-1, 1]$ on the analog computer. This set of equations still has the problem that the coefficients are really unwieldy for an analog computer. Thus, equations (16), (17), and (18) are scaled again with



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a common $\lambda_{\text{global}} = \frac{1}{100}$ yielding the following set of coupled differential equations that can now be implemented directly on an analog computer as shown in figure 1:¹

$$\begin{aligned}\dot{x} &= 0.533y - 0.4x + 0.4xz \\ \dot{y} &= 0.4125x + 0.2y - 1.875xz \\ \dot{z} &= 0.0183z + 1.2xy - 0.585x^2\end{aligned}$$

The system yields really beautiful phase space plots as shown in figures 2, 3, and 4. A pseudo-3D-projection is shown in figure 5 where y was plotted against $-0.5(x + z)$. These pictures were taken with a Canon EOS 50D camera mounted on a tripod in front of a LEADER LBO-9C oscilloscope with a makeshift cardboard box for shielding ambient light. The settings were ISO 400 with an exposure time of three seconds.

References

[LETELLIER et al. 2007] CHRISTOPHE LETELLIER, ROBERT GILMORE, "Topological analysis of a toroidal Lorenz-like attractor", <https://www.physics.drexel.edu/~bob/Papers/LorLi3.pdf>, retrieved 18.04.2025

¹The \approx has been replaced by $=$ to simplify the notation.

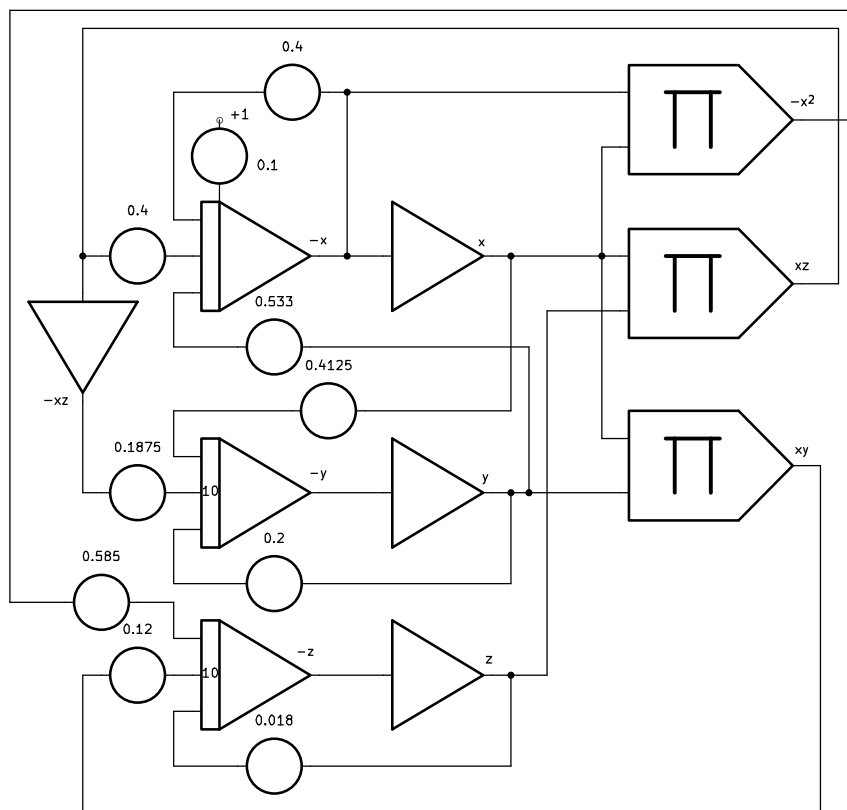


Figure 1: Analog computer program for the chaotic LI system

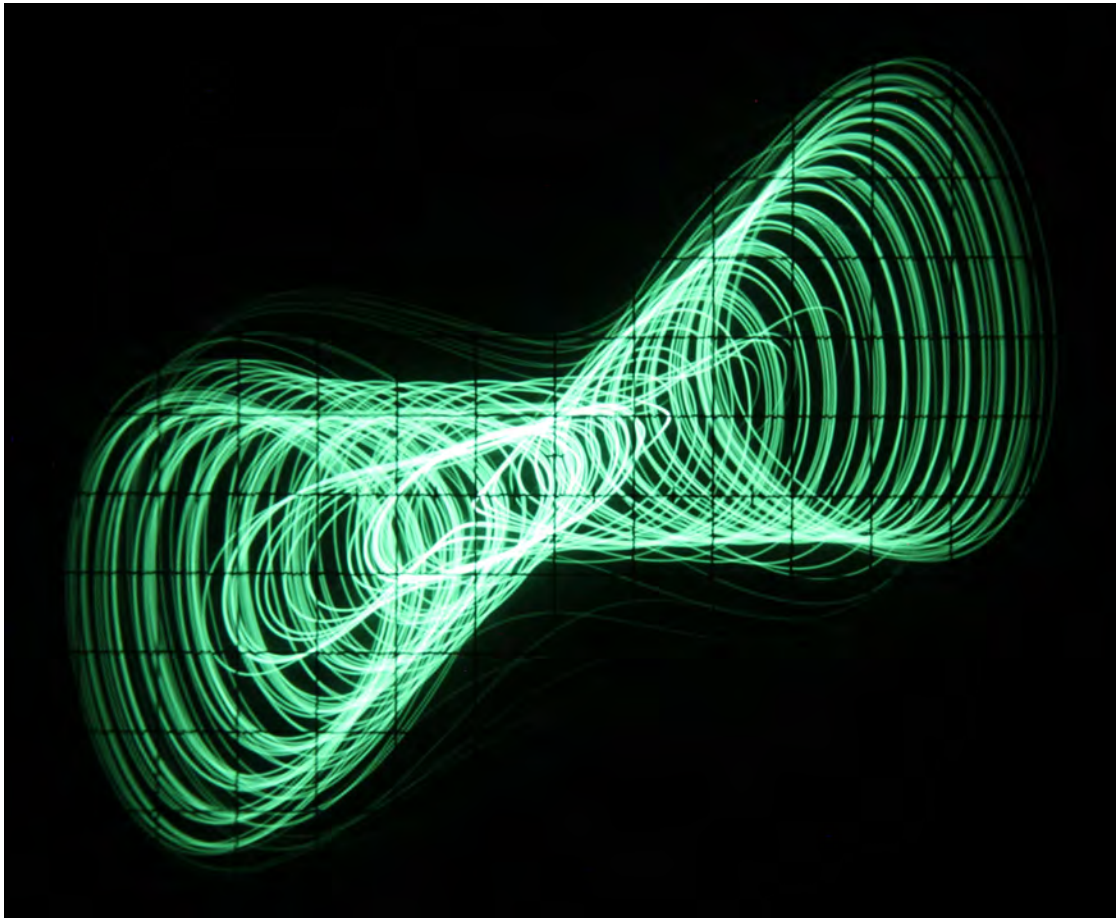


Figure 2: x/y phase space plot of the LI system

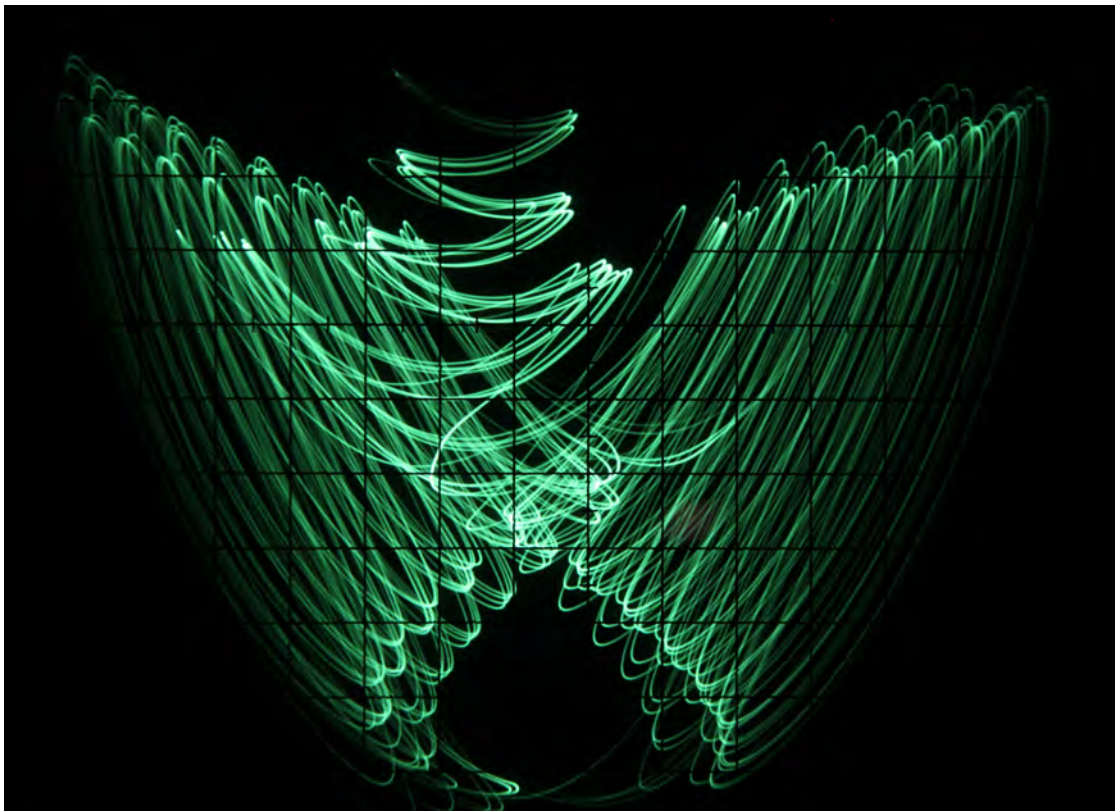


Figure 3: x/z phase space plot of the LI system

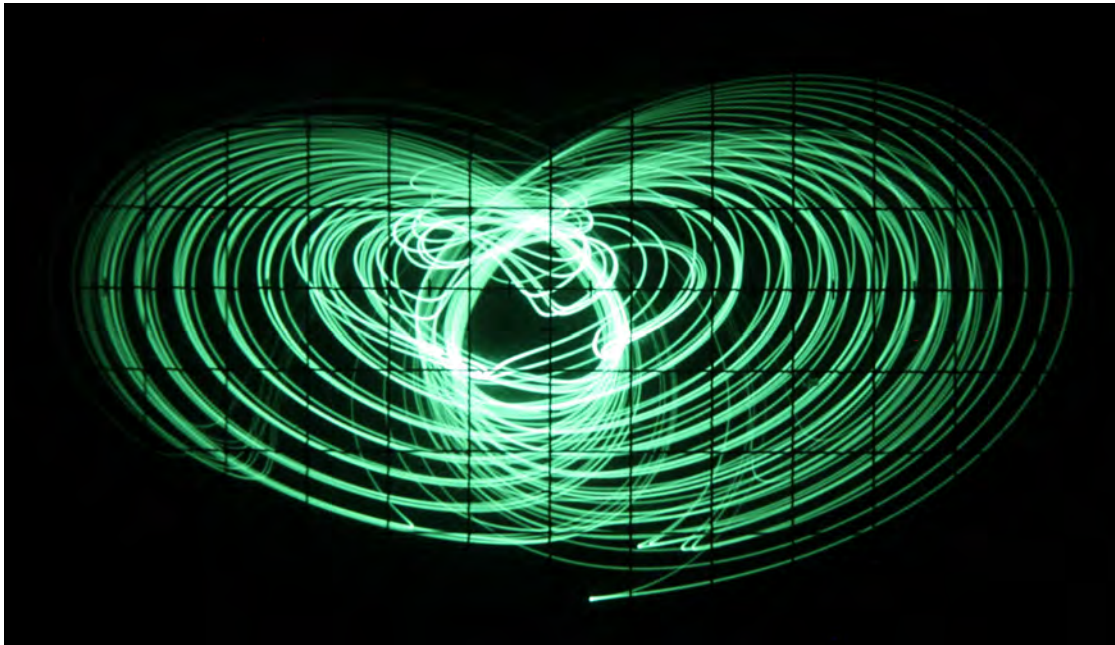


Figure 4: y/z phase space plot of the LI system

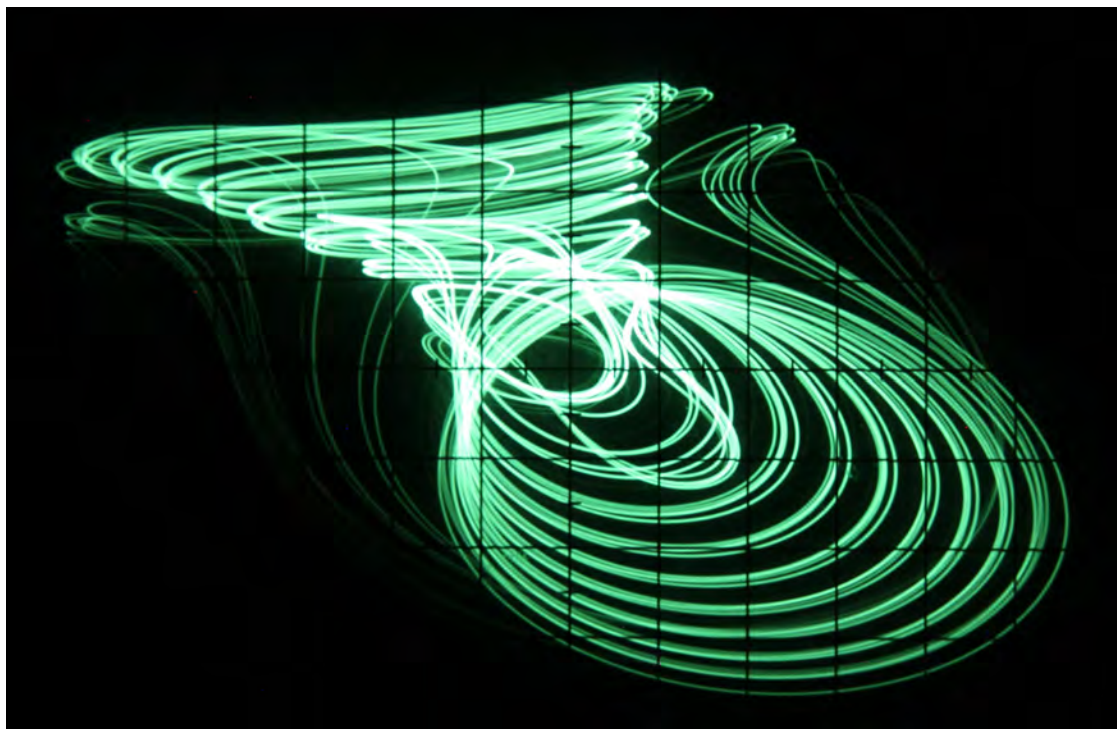


Figure 5: $y/ - 0.5(x + z)$ phase space plot of the LI system