Solving the two-dimensional heat-equation

1 Introduction

The aim of this application note is to solve the two-dimensional heat-equation

$$\dot{u} = \alpha \nabla^2 u$$

i. e.

$$\dot{u} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1}$$

on an analog computer. Here, α is the diffusity of the medium and is assumed to be $\alpha = 1$ in the following to simplify things (and save coefficient potentiometers in the setup). Discretizing equation (1) yields the following general difference quotient term for the individual nodes of the plane being simulated:

$$\dot{u}_{i,j} = \alpha \left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} \right) + q_{i,j}. \tag{2}$$

Here, $q_{i,j}$ denotes the heat injected or removed at/from node i, j. In the following, only $q_{0,0} \neq 0$. This approach essentially uses a finite difference method for solving the PDEs.

In the following example, a plate, 8×8 cm, which is insulated on its top and bottom surfaces is considered. The edges of this plate are held at a fixed temperature T. As this example exhibits symmetry along the x-, y-axes as well as along the 45 degree diagonals, only one octant of the problem has to be simulated, thereby saving a considerable number of computing elements, but at the cost of having to set up additional
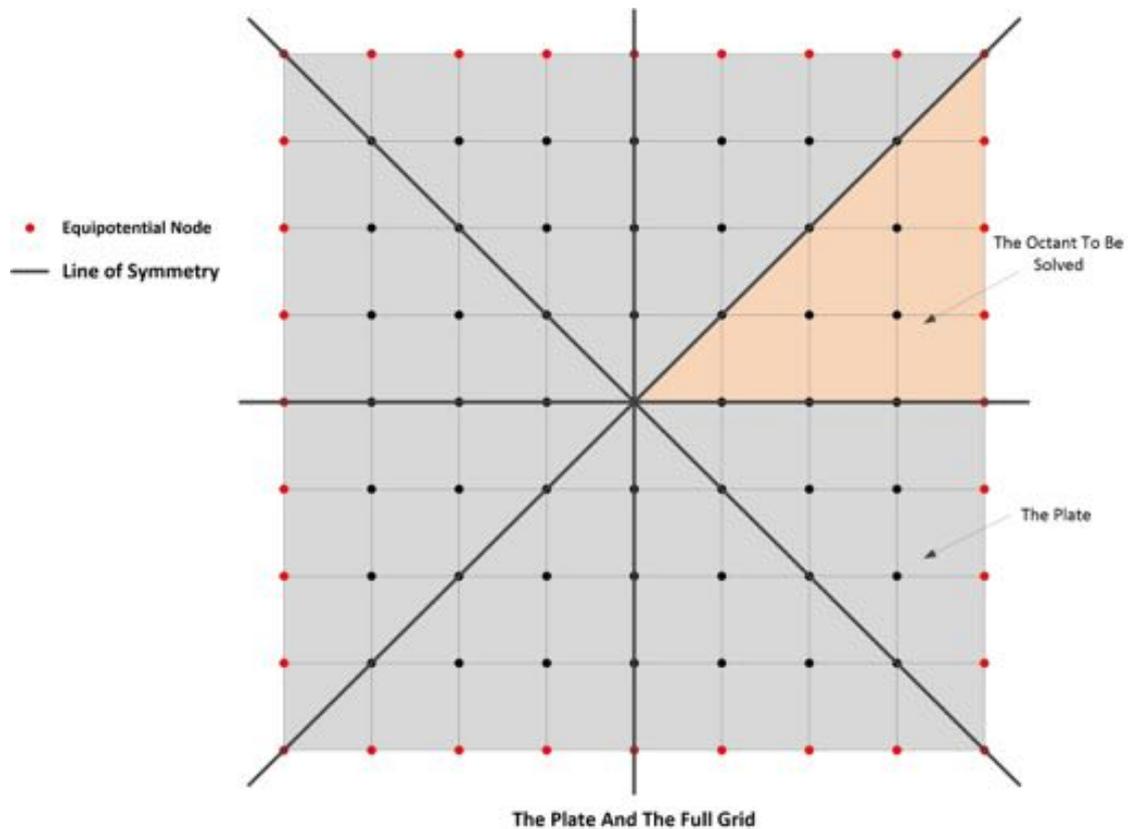
Figure 1: The heat conducting plate

boundary conditions. Figure 1 shows the 1×1 cm grid arrangement and the axes of symmetry.

2 Implementation

According to equation (2), every integrator representing the plate has five inputs: The output values of its direct neighbours in x- and y-
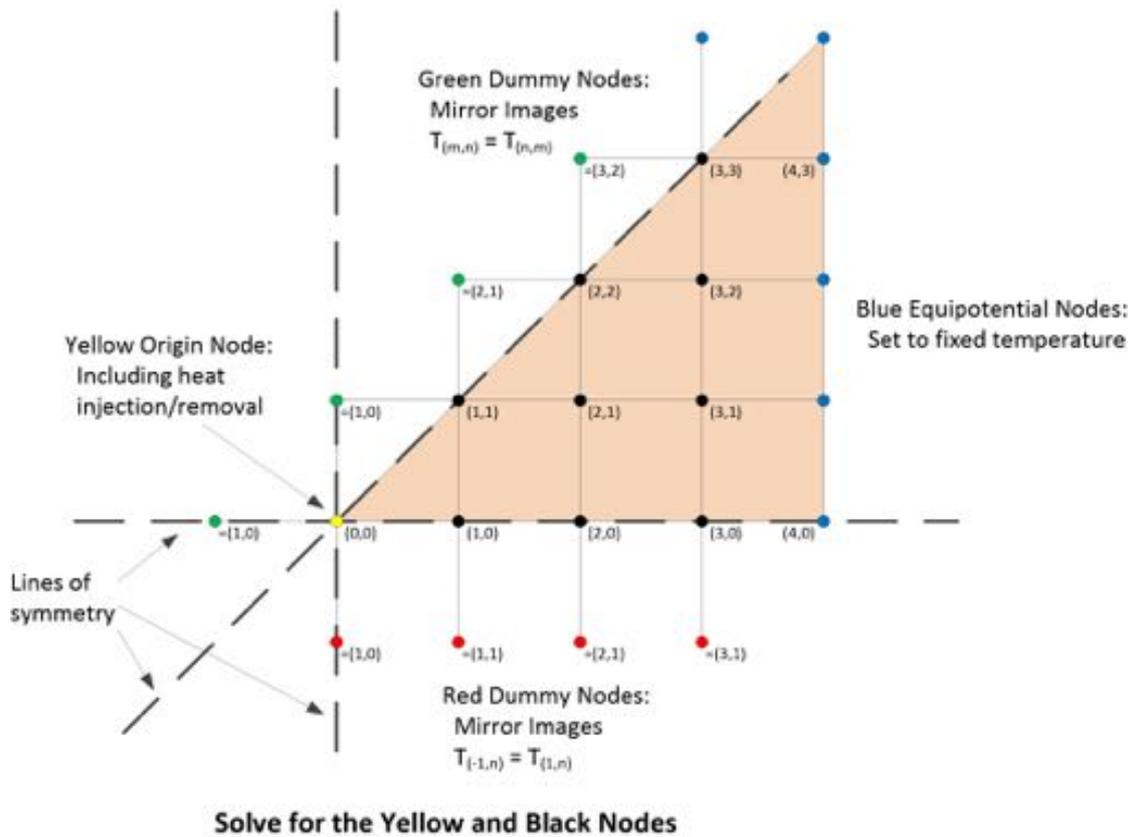
Figure 2: Solution of the 2-D problem

direction as well as its own (inverted) output for the $-4u_{i,j}$ term. An exception to this is node $0,0$ which has an additional input for the heat injected to the plate. To satisfy the symmetry, dummy nodes outside the octant, which are reflections in the lines of symmetry of nodes inside the octant are created. This is shown in figure 2.

The resulting schematic is shown in figure 3. It requires ten integrators, ten inverters (summers), and twelve potentiometers. The only drawback is that the integrators at the edges of the octant have many paralleled inputs with weight 1. These can be eliminated by substituting inputs with weight 10 and additional dedicated potentiometers as shown in figure 4.

3   Results

Figure 5 shows two typical results of this simulation. In all cases, node $u_{0,0}$ is subject to a step impulse $q_{0,0}$. The picture on the left shows $q_{0,0}$ and the outputs of nodes $u_{0,0}, u_{1,0}, u_{2,0}$. The picture on the right shows the outputs of the nodes along the diagonal – please note that the y-sensitivity of the oscilloscope has been doubled, compared with the picture on the left, to get a reasonable output.

Figure 6 again shows the impulse $q_{0,0}$ and the outputs of the first three nodes along the x-axis but this time with a boundary temperatur $T = 1$. It can be clearly seen that the nodes get hotter from the boundary to the middle of the plate while the heat spike introduced at node $u_{0,0}$ accelerates the heating of its neighbour nodes.
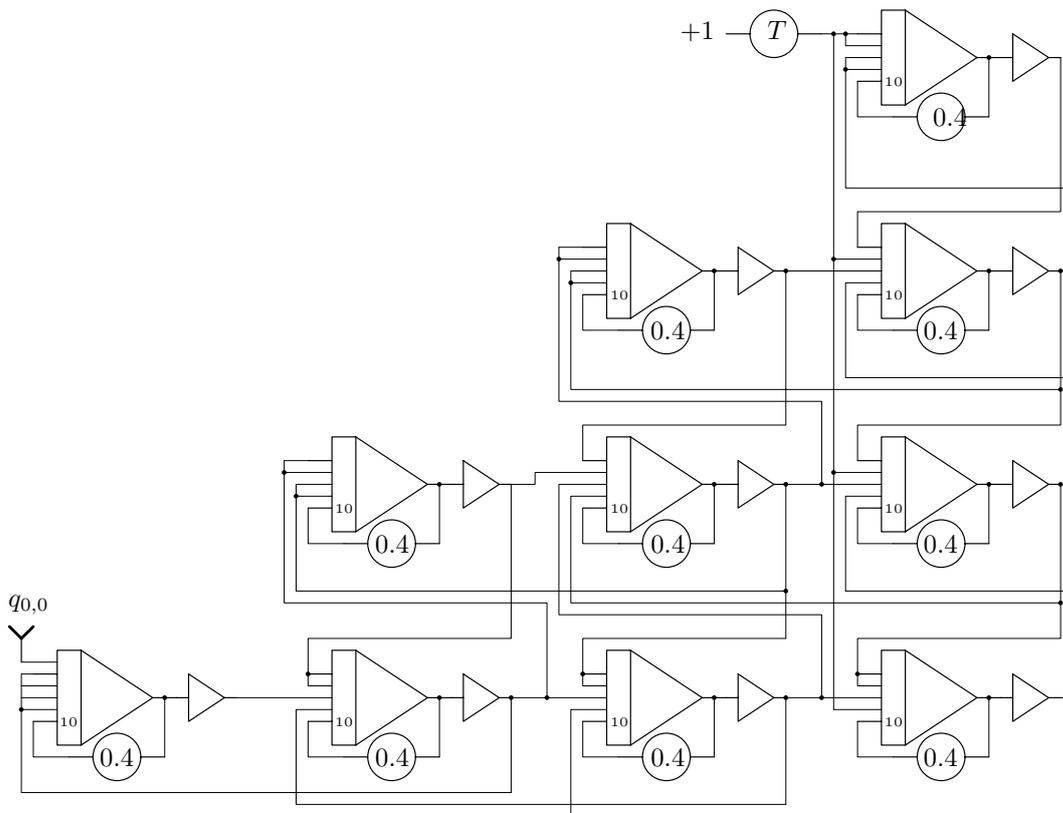
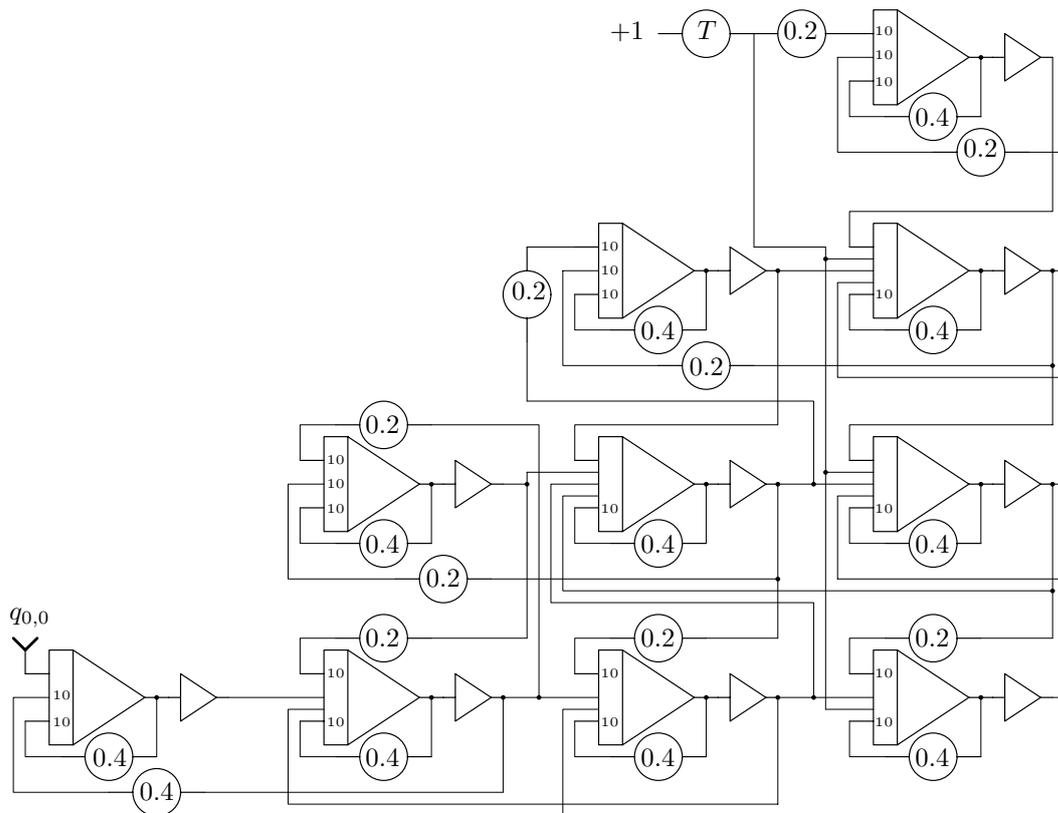Figure 3: Basic setup of the 2-dimensional heat problem

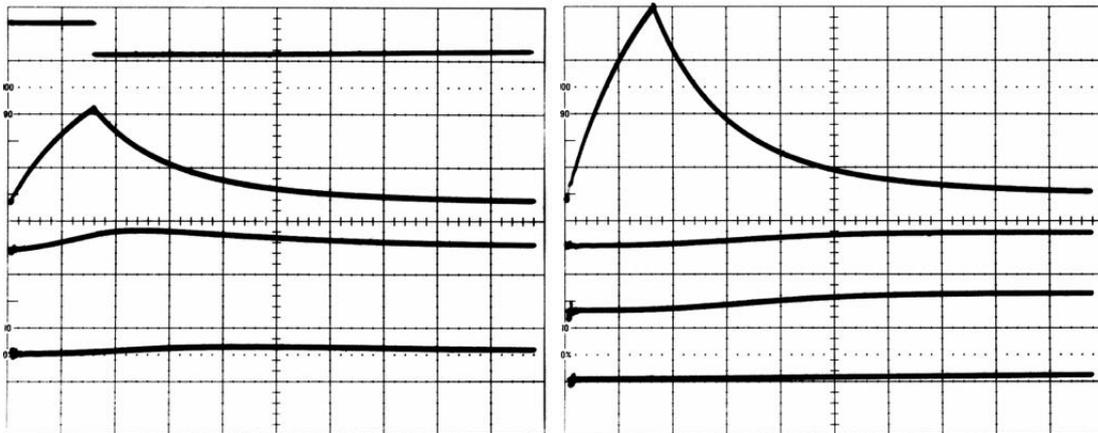Figure 4: Actual implementation of one octant

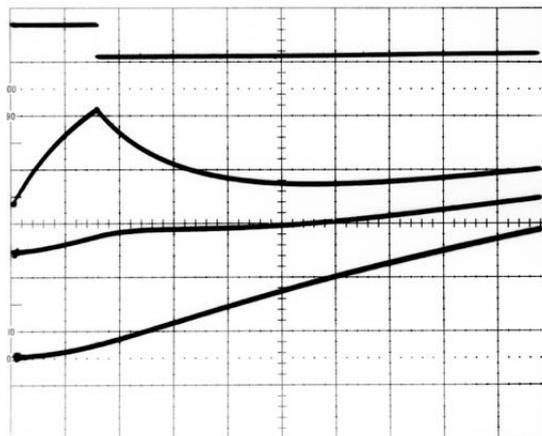Figure 5: Typical results with $T = 0$ at the plate boundary

Figure 6: Results along the x-axis with non-zero boundary temperature T