

Figure 1: Configuration of the inverted pendulum

Simulating an inverted pendulum

This application note describes the simulation of an inverted pendulum, i. e. a pendulum that has its mass above its pivot point which is mounted on a cart capable of moving along the x -axis as shown in figure 1. The aim is to keep this pendulum in its upright position by controlling a force F acting on the cart in order to move it from left to right and vice versa.



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1 Equations of motion

If such a pendulum were mounted on a stationary cart, its equation of motion would look the same as that for a mathematical pendulum, i. e.

$$\ddot{\varphi} - \frac{g}{l} \sin(\varphi) = 0$$

which does not cover the more complex example of an inverted pendulum mounted on a cart with one degree of freedom. To derive the equations of motions for this more complex setup, the Lagrangian

$$L = T - V \quad (1)$$

is used, where T represents the total kinetic energy, while V is the potential energy of the system.¹ The total potential energy is

$$V = mgl \cos(\varphi), \quad (2)$$

while the total kinetic energy is the sum of the kinetic energies of the pendulum bob having mass m and the moving cart with mass M , i. e.

$$T = \frac{1}{2} (Mv_c^2 + mv_p^2), \quad (3)$$

where v_c is the velocity of the cart, while v_p is the velocity of the pendulum bob. Obviously,

$$v_c = \dot{x} \quad (4)$$

and (a bit less obviously)

$$v_p = \sqrt{\left(\frac{d}{dt}(x - l \sin(\varphi))\right)^2 + \left(\frac{d}{dt}(l \cos(\varphi))\right)^2}.$$

The left term under the square root is

$$(\dot{x} - l\dot{\varphi} \cos(\varphi))^2 = \dot{x}^2 - 2\dot{x}l\dot{\varphi} \cos(\varphi) + l^2\dot{\varphi}^2 \cos^2(\varphi),$$

¹The following derivation follows basically https://en.wikipedia.org/wiki/Inverted_pendulum, retrieved March 29th, 2019.

while the right term is

$$(-l\dot{\varphi} \sin(\varphi))^2 = l^2\dot{\varphi}^2 \sin^2(\varphi),$$

yielding

$$\begin{aligned} v_p &= \dot{x}^2 - 2\dot{x}l\dot{\varphi} \cos(\varphi) + l^2\dot{\varphi}^2 \cos^2(\varphi) + l^2\dot{\varphi}^2 \sin^2(\varphi) \\ &= \dot{x}^2 - 2\dot{x}\dot{\varphi}l \cos(\varphi) + l^2\dot{\varphi}^2 \end{aligned} \quad (5)$$

since $\sin^2(\varphi) + \cos^2(\varphi) = 1$.

Substituting (2) and (3) into (1) considering (4) and (5) yields the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}(\dot{x}^2 - 2\dot{x}\dot{\varphi}l \cos(\varphi) + l^2\dot{\varphi}^2) - mgl \cos \varphi \\ &= \frac{1}{2}(M + m)\dot{x}^2 + m\dot{x}\dot{\varphi}l \cos(\varphi) + \frac{1}{2}ml^2\dot{\varphi}^2 - mgl \cos(\varphi). \end{aligned} \quad (6)$$

The EULER-LAGRANGE-equations of the general form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

yield

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} &= F \text{ and} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} &= 0 \end{aligned}$$

with the generalized coordinates $q_1 = x$ and $q_2 = \varphi$. These yield

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial \dot{x}} &= (M + m)\dot{x} + m\dot{\varphi}l \cos(\varphi) \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) &= (M + m)\ddot{x} + m\ddot{\varphi}l \cos(\varphi) - m\dot{\varphi}l \sin(\varphi) \end{aligned}$$

and

$$\begin{aligned}\frac{\partial L}{\partial \varphi} &= ml\dot{x}\dot{\varphi} \sin(\varphi) + mgl \sin(\varphi) \\ \frac{\partial L}{\partial \dot{\varphi}} &= -ml\dot{x} \cos(\varphi) + ml^2\dot{\varphi} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) &= -ml\ddot{x} \cos(\varphi) + ml\dot{x}\dot{\varphi} \sin(\varphi) + ml^2\ddot{\varphi}\end{aligned}$$

with (6).

These finally yield

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = (M + m)\ddot{x} - ml\ddot{\varphi} \cos(\varphi) + ml\dot{\varphi}^2 \sin(\varphi) = F \quad (7)$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) = -ml\ddot{x} \cos(\varphi) + ml\dot{x}\dot{\varphi} \sin(\varphi) + ml^2\ddot{\varphi} - ml\dot{x}\dot{\varphi} \sin(\varphi) - mgl \sin(\varphi) = 0.$$

Dividing that last equation by ml yields

$$l\ddot{\varphi} - \ddot{x} \cos(\varphi) - g \sin(\varphi) = 0$$

which can be solved for $\ddot{\varphi}$ yielding

$$\ddot{\varphi} = \frac{1}{l} (\ddot{x} \cos(\varphi) + g \sin(\varphi)). \quad (8)$$

The two equations (7) and (8) fully describe the motion of an inverted pendulum mounted on a cart capable of moving along the x direction under an external force F and can now be used to derive an analog computer setup for this problem.

Assuming that the mass m of the pendulum bob is negligible compared to the mass M of the moving cart, (7) can be simplified to $M\ddot{x} = F$, i. e. the movement of the pendulum mounted on the cart has no relevant effect on the cart's movement. Under this assumption, the inverted pendulum problem can be described by (8) alone with

$$\ddot{x} = \frac{F}{M}.$$



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2 Computer setup

Equation 8 can be directly converted into a corresponding analog computer program as shown in figure 2. Based on the input variable \ddot{x} , this setup generates the time derivative of the generalized coordinate $\dot{\varphi}$ since $\dot{\varphi}$ is used to generate the two required harmonic terms $\sin(\varphi)$ and $\cos(\varphi)$ without the need for a sine/cosine function generator as shown in figure 3. Note that the term $\frac{1}{l}$ has been deliberately omitted by defining $l := 1$.² γ_1 controls how much the acceleration \ddot{x} affects the pendulum while γ_2 controls its effect on the movement of the cart.

The potentiometer labelled g yields the gravitational acceleration factor while β is an extension of the basic equation of motion and introduces a damping term for the angular velocity $\dot{\varphi}$ of the pendulum, making the simulation more realistic. Setting $\beta = 0$ results in a frictionless pendulum.

The two integrators in series followed by an inverter yield $-x$ based on the input variable \ddot{x} which is used in the following to display the pendulum mounted on its moving cart.

Generating the terms $\sin(\varphi)$ and $\cos(\varphi)$ is done with the circuit shown in figure 3 which basically solves a differential equation of the form $\ddot{y} = -\dot{\varphi}^2 y$.

What would be an analog computer setup without a proper visualization of the dynamic system? The program shown in figure 4 displays the cart with its attached pendulum rod as it moves along the x -axis, controlled by the input function \ddot{x} . For a little manual demonstration, a double pole switch with neutral middle position can be used to generate a suitable input signal \ddot{x} .

To generate a flicker-free display, a rather high frequency amplitude stabilized quadrature signal pair $\sin(\omega t)$ and $\cos(\omega t)$ is required. A typical circuit yielding such a signal pair can be found in [ULMANN 2017, p. 67]. This signal is used to draw a circular or elliptical cart as well as the rotating pendulum rod.

The length of the rod is set by the potentiometer labelled l . Since the pendulum rod should have one of its ends fixed at the center of the cart figure, the $\sin(\omega t)$ -signal has to be shifted by some amount r , otherwise the display would not show a pendulum but

²The output function $\ddot{\varphi}$ will be used later.

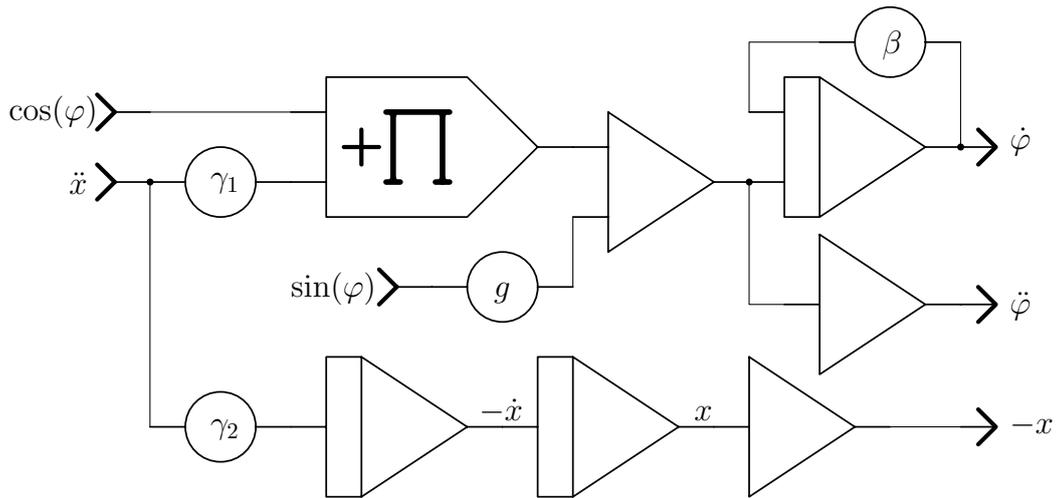


Figure 2: Motion of the pendulum mounted on its cart

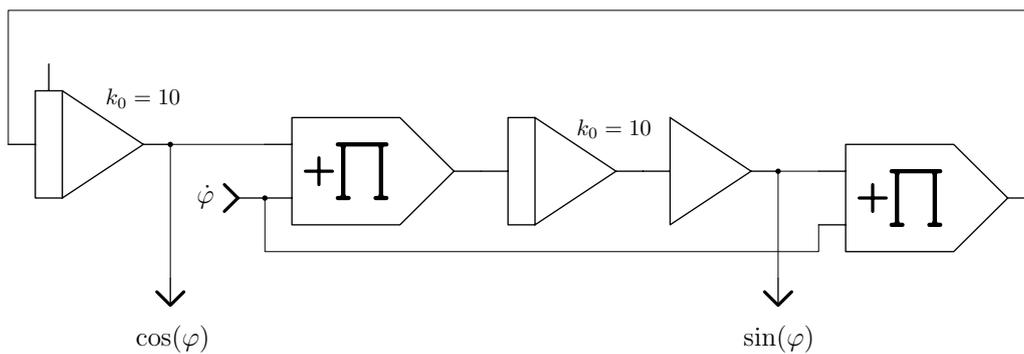


Figure 3: Generating $\sin(\varphi)$ and $\cos(\varphi)$

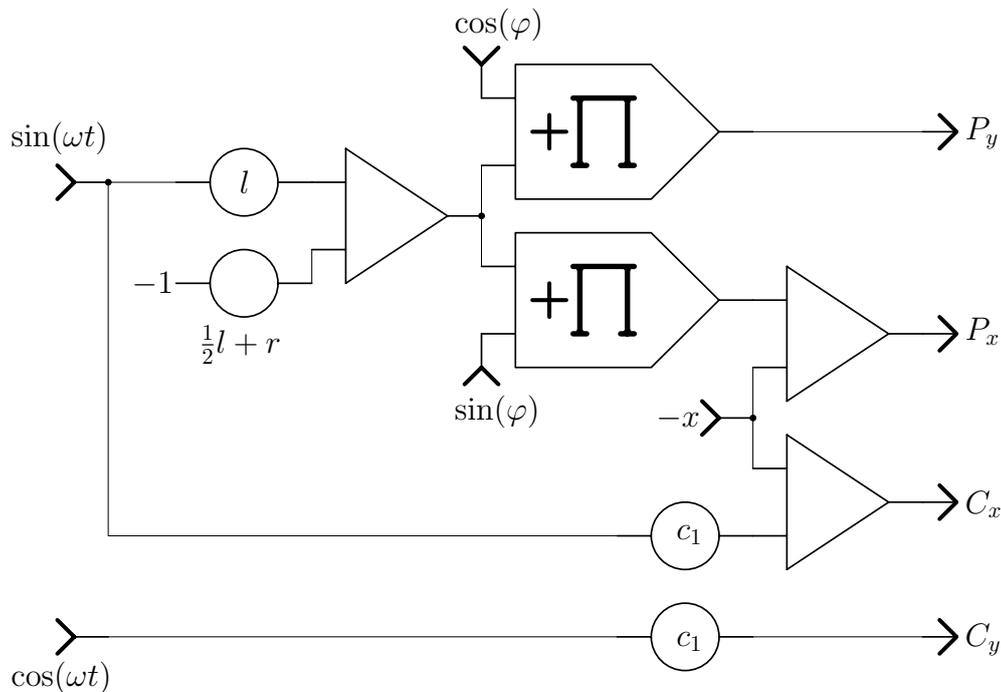


Figure 4: Display of the pendulum mounted on its cart

rod rotating around its center. The actual rotation is done by multiplication with $\sin(\varphi)$ and $\cos(\varphi)$. The circuit yields two output signal-pairs: P_x and P_y are used to draw the pendulum rod while C_x and C_y display the moving cart. Therefore, an oscilloscope with two independent (multiplexed) x/y -inputs is required for this simulation.

A screenshot of a typical resulting display with the pendulum just tipping over is shown in figure 5.

3 Heavy pendulum bob

But what happens when the mass of the pendulum bob is no longer negligible? In this case equation (7) can no longer be simplified as before but must be implemented to implement the effect of the swinging pendulum to the cart it is mounted on. Solving

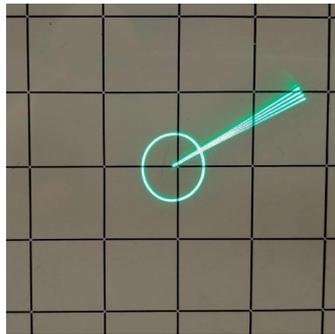


Figure 5: Display of the pendulum mounted on its cart

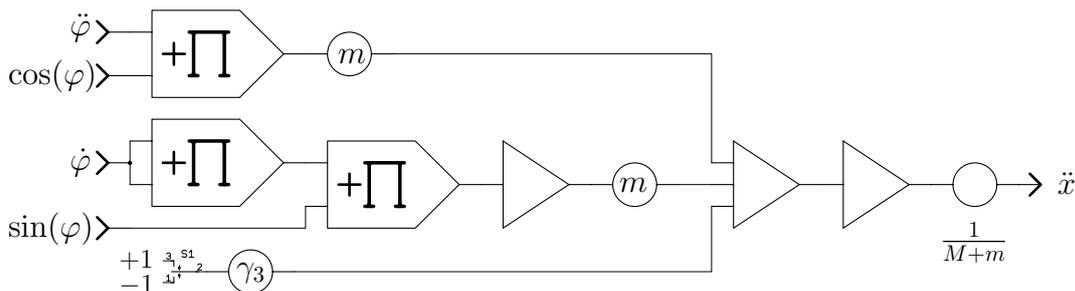


Figure 6: Effect of a heavy mass pendulum on the moving cart

this equation for \ddot{x} yields

$$\ddot{x} = \frac{1}{M+m} (F + ml\ddot{\varphi} \cos(\varphi) - ml\dot{\varphi}^2 \sin(\varphi))$$

which can be easily transformed into a computer setup as shown in figure 6.

The function \ddot{x} generated by this circuit is now fed into the associated input of the sub-circuit shown in figure 2. An external force acting on the cart can be introduced by means of switch S_1 while γ_3 controls the strength of this force.



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References

[ULMANN 2017] BERND ULMANN, *Analog Computer Programming*, Create Space, 2017